Eigen-vectors and Eigen-values.

Let A be a square matrix of n rows and columns and r a vector with n componets:

$$(1) \quad r = x_1 + x_2 + x_3 + \dots x_n$$

where for simplicity we have omitted the unit hyper vectors i_j.

When A is applied to just any vector p one generally gets another vector s, say, which has a different magnitude and direction than that of p:

$$(2) \quad A \cdot p = s.$$

We want to find a vector r so that A will not change its direction but at most change its size. In this case we write:

$$(3) \quad A \cdot r = k\, r.$$

Here r is called an eigen-vector of the matrix A and the scalar multiplier k is called an eigen-value of the matrix A.

One may write (3) in the form:

$$(4) \quad (A - k I) \cdot r = 0.$$

Equation (4) has non-trivial solutions for r only when:

$$(5) \quad (A - k I)_0 = 0.$$

Expanding equation (5) one gets:

$$(6) \quad k^n - T_1 K^{n-1} + T_2 k^{n-2} - T_3 k^{n-3} + \ldots (-1)^n T_n = 0.$$

where T_j is the sum of the determinants of order j down the main diagonal.

The number of these determinants obeys the binomial law according to Mutation Geometry. This knowledge saves the messy expansion of large determinants. For example, for a third o order determinant we would have:

T_0	T_1	T_2	T_3
1	3	3	1

which means one would have one determinant of the 0 th order, 3 of the first order, 3 of the second order, and 1 of the 3rd order. For a 4th order determinant we would have:

T_0	T_1	T_2	T_3	T_4
1	4	6	4	1

and in general the binomial law.

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$T_0 = 1$$

$$T_1 = a_{11} + a_{22} + a_{33}$$

$$T_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$T_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We do a numerical example of the third order, calculating the three eigen values and their corresponding eigen-vectors.

Numerical example: Given the matrix:

$$A = \begin{matrix} 2 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & -2 & 4 \end{matrix}$$

$$T_1 = 2 + 3 + 4 = 9$$

$$T_2 = \begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix} + \begin{matrix} 3 & -1 \\ -2 & 4 \end{matrix} + \begin{matrix} 2 & -1 \\ 3 & 4 \end{matrix} = 26$$

$$T_3 = 24.$$

$$k^3 - 9k^2 + 26k - 24 = 0.$$

$$(k - 2)(k - 3)(k - 4) = 0.$$

$$k_1 = 2, \quad k_2 = 3, \quad k_3 = 4.$$

To find the eigen-vector r_1 corresponding to k_1 we put k_1 into (4) and get for the first two rows:

$$\begin{matrix} 0 & 1 & -1 \\ 1 & 1 & -1 \end{matrix}$$

taking a gamma vector of this (column cofactors) we get:

$$r_1 = 0 \quad -1 \quad -1$$

In the same way we get for k_2 :

$$\begin{matrix} -1 & 1 & 1 \\ 1 & 0 & -1 \end{matrix}$$

and the gamma for this is:

$$r_2 = -1 \quad +0 \quad -1$$

For k_3:

$$\begin{matrix} -2 & +1 & -1 \\ 1 & -1 & -1 \end{matrix}$$

The gamma for this is:

$$r_3 = -2 \quad -3 \quad +1 .$$

$$A = \begin{matrix} 2 & +1 & -1 \\ 1 & +3 & -1 \\ 3 & -2 & +4 \end{matrix}$$

$$k_1 = 2, \quad r_1 = 0 \quad - 1 \quad - 1$$

$$k_2 = 3, \quad r_2 = 1 \quad + 2 \quad + 1$$

$$k_3 = 4, \quad r_3 = -2 \quad - 3 \quad + 1$$

In this demonstration we have calculated the eigen-values first then used them to easily get the corresponding eigen-vectors. Now suppose want to calculate the eigen-vectors first and then use them to get the eigen-values. We may write our eigen equation as

$$(7) \qquad A \cdot r^1 = k\, r^1$$

where we have cancelled the magnitude r_0 from each side of the eigen-equation. Multiply both sides of the eigen-equation (7) by r^1 and get:

$$(8) \qquad r^1 \cdot A \cdot r^1 = k\, r^1 \cdot r^1 = k$$

If we can find r^1 then equation (8) will give k, the eigen value. Multiply equation (7) by $\breve{r}$ and get:

$$(9) \qquad \breve{r} \cdot A \cdot r = 0$$

From Mutation Geometry we have the identities:

$$(10) \qquad r_3 = (i_1 \cdot r)\, i_2 - (i_2 \cdot r)\, i_1$$

$$(11) \qquad r_2 = (i_1 \cdot r)\, i_3 - (i_3 \cdot r)\, i_1$$

$$(12) \qquad r_1 = (i_2 \cdot r)\, i_3 - (i_3 \cdot r)\, i_2$$

Multiply equation (7) by equations (10) , (11) and (12) and get:

$$(13) \qquad r \cdot (i_1 (a_{21} i_1 + a_{22} i_2 + a_{23} i_3) - i_2 (a_{11} i_1 + a_{12} i_2 + a_{13} i_3)) \cdot r = 0$$

$$((14) \qquad r \cdot (i_1 (a_{31} i_1 + a_{32} i_2 + a_{33} i_3) - i_3 (a_{11} i_1 + a_{12} i_2 + a_{13} i_3)) \cdot r = 0$$

$$(15) \quad r \cdot (i_2 (a_{31} i_1 + a_{32} i_2 + a_{33} i_3) - i_3(a_{21} i_1 + a_{22} i_2 + a_{23} i_3)) \cdot r = 0.$$

One of these is redundant, say (15), for it can be obtained from the other two. We write our eigen-vector as:

$$(16) \quad r = x_1 (i_1 + h_2 i_2 + h_3 i_3)$$

Put equation (16) into (13) and (14) and get:

$$(17) \quad a_{21} + a_{22} h_2 + a_{23} h_3 - h_2(a_{11}+ a_{12}h_2+ a_{13}h_3)$$

$$(18) \quad a_{31} + a_{32} h_2 + a_{33} h_3 - h_3(a_{11}+ a_{12}h_2+ a_{13}h_3)$$

We thus arrive at two quadratics in h_2 and h_3. For our matrix:

$$A = \begin{matrix} 2 & + & 1 & - & 1 \\ 1 & + & 3 & - & 1 \\ 3 & - & 2 & + & 4 \end{matrix}$$

the two quadratics become:

$$(19) \quad 1 + 3 h_2 - h_3 = h_2(2 + h_2 - h_3)$$

$$(20) \quad 3 - 2 h_2 + 4 h_3 = h_3(2 + h_2 - h_3).$$

By inspection , in this simple case, one set of values is:

$$h_2 = 2$$
$$h_3 = 1$$

then

$$r^1 = (i_1 + 2 i_2 + i_3) / \quad 6$$

$$k = r^1 . A . r^1 = 3$$

which is one of the eigen-values already obtained by the first method. Another set of values of (19) and (20) is:

$$h_2 = 3/2$$
$$h_3 = -1/2$$
$$r = (i_1 + 3/2 \, i_2 - 1/2 \, i_3) = (2 + 3 - 1)$$

We shall find the corresponding k in a slightly different way just fot variety. We write:

$$A \cdot r = k\,r$$

$$A \cdot r = 8 + 12 - 4 = 4\,(2 + 3 - 1) = 4\,r = k\,r$$

so $k = 4$.

When x_1 is 0 our eigen-vector cannot be written :

$r = x_1\,(1 + h_2 + h_3)$. Instead one may write:

$$(21) \quad r = x_2\,(h_1 + 1 + h_3) \quad \text{or}$$

$$(22) \quad r = x_3\,(h_1 + h_2 + 1).$$

Using equation (23) one finds, as one set:

$$h_1 = 0$$

$$h_2 = 1$$

$$r = 0 + 1 + 1$$

$$A \cdot r = 0 + 2 + 2 = 2\,(0 + 1 + 1) = 2\,r$$

So $k = 2$. The three eigen-values corresponding to our three eigen-vectors are:

$$r_1 = 0 + 1 + 1. \qquad k_1 = 2$$

$$r_2 = 1 + 2 + 1, \qquad K_2 = 3$$

$$r_3 = 2 + 3 - 1, \qquad k_3 = 4$$

which agree with the previous calculation. We now write the generalization for n dimensions:

$$A = \begin{matrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{matrix}$$

$$r = x_1\,(1 + h_2 + h_3 + \ldots h_n).$$

$$a_{21} + a_{22} h_2 + \dots a_{2n} h_n = h_2(a_{11} + a_{12} h_2 + \dots a_{1n} h_n)$$

$$(23) \quad a_{31} + a_{32} h_2 + \dots a_{3n} h_n = h_3(\dots\dots\dots\dots)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1} + a_{n2} h_2 + \dots a_{nn} h_n = h_n(a_{11} + a_{12} h_2 + \dots a_{1n} h_n))$$

The system of quadratic equations (23) is a pioneering one from the New Science of Mutation Geometry.

When the system is large these quadratics lend themselves easily to numerical solutions.

We shall call this scheme of calculating eigen-vectors and eigen-values the H - Way, and that in (6) the T - Way.

For large systems the H - Way seems the more tractible. For small systems there does not seem much difference. Experience will help one to decide, in either case.