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Given:

(1) $$\varphi \cdot r = R$$

where r and R are known, to find φ.

Let R be the column vector

(2) $$R = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix}$$

and

(3) $$r = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Then

$$(4) \quad \varphi = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

Then (1) may be written in the form:

$$(5) \quad \psi \cdot L = R$$

where L is the column vector:

$$(6) \quad L = \begin{vmatrix} a_{11} \\ a_{12} \\ \cdots \\ a_{1n} \\ a_{21} \\ a_{22} \\ \cdots \\ a_{2n} \\ \cdots \\ a_{m1} \\ a_{m2} \\ \cdots \\ a_{mn} \end{vmatrix}, \quad \psi = \begin{vmatrix} x_1 & x_2 \cdots x_n & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & x_1 & x_2 \cdots x_n & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 \cdots x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & x_n \end{vmatrix}$$

to m rows of the above pattern

example

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix} = \begin{vmatrix} 1 \\ 5 \end{vmatrix}$$

$$b = \begin{vmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{vmatrix}, \quad \varphi = \begin{vmatrix} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{vmatrix}$$

$$(7) \qquad \begin{vmatrix} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{vmatrix} = \begin{vmatrix} 1 \\ 5 \end{vmatrix}$$

Multiplying the first row by 5 and subtracting the second row one gets:

$$(8) \quad \begin{vmatrix} 5 & -5 & 10 & -1 & 1 & -2 \end{vmatrix} \begin{vmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{vmatrix} = 0$$

This eq. is satisfied if we choose $a_{11} = 1$ and $a_{21} = 5$ and all the other a's zero. q then becomes

$$q = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \end{vmatrix}$$

Any other matrix which when multiplied by $r = 0$ may be added to q. they are

$$a_{12} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad a_{13} \begin{vmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$a_{22} \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \qquad a_{23} \begin{vmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

and

$$h_1 \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \quad h_2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\varphi = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \end{vmatrix} + a_{12} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} +$$

$$a_{22} \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} + a_{13} \begin{vmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \end{vmatrix} +$$

$$a_{23} \begin{vmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \end{vmatrix} + h_1 \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix} +$$

$$h_2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

Initially one could have chosen any two of the as which would satisfy eq. (8) then proceed from there.