Any system of linear equations

$$(1)\quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots\ldots\ldots a_{1n}x_n = p_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \ldots\ldots\ldots a_{2n}x_n = p_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad . \quad . \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 \ldots\ldots\ldots a_{nn}x_n = p_n \end{array}$$

may be factored into

$$(2)\quad \begin{array}{l} a_1 \cdot r = p_1 \\ a_2 \cdot r = p_2 \\ \ldots\ldots \quad . \quad . \\ a_n \cdot r = p_n \end{array}$$

where

$$(3)\quad \begin{array}{l} a_1 = a_{11}i_1 + a_{12}i_2 + a_{13}i_3 \ldots\ldots \quad a_{1n}i_n \\ a_2 = a_{21}i_1 + a_{22}i_2 + a_{23}i_3 \ldots\ldots \quad a_{2n}i_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n = a_{n1}i_1 + a_{n2}i_2 + a_{n3}i_3 \ldots\ldots \quad a_{nn}i_n \end{array}$$

$$(4)\quad r = x_1 i_1 + x_2 i_2 + x_3 i_3 \ldots\ldots \quad x_n i_n$$

where i_1 , i_2 , ... i_n is a set of unit orthogonal hyper vectors.

By performing the multiplication indicated in (2) with the factors in (3) and (4) set in the result is (1).

A formal solution to (2) and hence (1) may be written:

$$(5)\quad r = p_1 \hat{a}_1 + p_2 \hat{a}_2 + p_3 \hat{a}_3 + \ldots\ldots p_n \hat{a}_n .$$

where

$$(6)\quad (\hat{a}_1 , \hat{a}_2 , \hat{a}_3 \ldots\ldots \hat{a}_n)$$

is the reciprocal system to

(7) $(a_1 , a_2 , a_3 \cdots a_n)$

Here

(8) $$a_i \cdot \widehat{a}_j = 0 \qquad i \neq j$$

(9) $$a_j \cdot \widehat{a}_j = 1$$

If one multiplies (5) by a_i one gete

(10) $$a_i \cdot r = p_i$$

which shows that (5) is a solution of (2) and hence (1).

It may be shown that the slurred quantities $\widehat{a}_i$ etc are:

(11)
$$\begin{aligned}
\widehat{a}_1 &= (a_2 \times a_3 \times a_4 \cdots \times a_n) / a_1 \cdot (a_2 \times a_3 \times a_4 \cdots a_n) \\
\widehat{a}_2 &= (a_3 \times a_4 \times a_5 \cdots \times a_1) / a_1 \cdot (a_2 \times a_3 \times a_4 \cdots \times a_n) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\widehat{a}_n &= (a_1 \times a_2 \times a_3 \cdots \times a_{n-1}) / a_1 \cdot (a_2 \times a_3 \times a_4 \cdots \times a_n)
\end{aligned}$$

Our multiplication signs $\cdot$ and $\times$ are signs of vector multiplication, giving a scalar and vector respectively. The dem nominators in (11) will never need to be calculated as such.

The components of the numerator part of a_1 turn out to be the column cofactors in order from left to right in the system

(12)
$$\begin{array}{c} a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_n \end{array}$$

The components of the numerator part of a_2 turn out to be proportional to the column cofactors in order from left to right in the system

$$
(13) \qquad \begin{array}{l} a_1 \\ a_3 \\ \cdot \\ \cdot \\ a_n \end{array}
$$

The components of the numerator part of a_i turn out to be proportional to the column cofactors in order fron left to right in the system

$$
(14) \qquad \begin{array}{l} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{i-1} \\ a_{i+1} \\ a_{i+2} \\ \cdot \\ \cdot \\ a_n \end{array}
$$

With this preliminary view established we (for speed and power) rewrite (2) in the form:

$$
(15) \qquad \begin{array}{lcl} a_1 \cdot r & = & p_1 \\ A_2 \cdot r & = & 0 \\ A_3 \cdot r & = & 0 \\ \cdots\cdots & & \\ A_n \cdot r & = & 0 \end{array}
$$

where

$$
(16) \qquad \begin{array}{l} A_2 = p_1 a_2 - p_2 a_1 \\ A_3 = p_1 a_3 = p_3 a_1 \\ \cdots\cdots\cdots\cdots \\ A_n = p_1 a_n - p_n a_1 \end{array}
$$

A solution to (15) is

$$(17) \qquad r = p_1 \,\widehat{a}_1$$

The components of a_1 are proportional to the column co-factors in order from left to right in the system

$$(18) \qquad \begin{matrix} A_2 \\ A_3 \\ \cdot \\ \cdot \\ A_n \end{matrix}$$

and so the components of r ($x_1, x_2, x_3 \ldots, x_n$) are proportional to these column cofactors. Call these column cofactors :

$$(19) \qquad (h M_1, h M_2, h M_3 \ldots h M_n)$$

To get the proportionality factor S one substitutes the the M_1 , M_2 , $M_3 \ldots M_n$ into equation one of (15) for the components of r and gets

$$(20) \qquad S = p_1 / (a_{11} M_1 + a_{12} M_2 + a_{13} M_3 \ldots a_{1n} M_n).$$

Thus

$$(21) \qquad r = S (M_1 , M_2 , M_3 \ldots , M_n).$$

We claim that equation (21) is the ultimate in simplicity for system solutions.

We shall do a number of illustrative examples to see the new theory in action.

Example 1

Solve the system:

$$(1) \qquad \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 2 \\ 3 & -1 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 8 \\ 6 \\ 2 \end{vmatrix}$$

This may be written

$$(2) \qquad \begin{vmatrix} 2 & 1 & 1 \\ 2 & 11 & -5 \\ -2 & 1 & 1 \end{vmatrix} \begin{vmatrix} x \\ x \\ x \end{vmatrix} = \begin{vmatrix} 8 \\ 0 \\ 0 \end{vmatrix}$$

The column cofactors of the second and third rows are:

$$h\ M = \begin{vmatrix} 11 & -5 \\ 1 & 1 \end{vmatrix} = (8)(2)$$

$$h\ M = -\begin{vmatrix} 2 & -5 \\ -2 & 1 \end{vmatrix} = (8)(1)$$

$$h\ M = \begin{vmatrix} 2 & 11 \\ -2 & 1 \end{vmatrix} = (8)(3)$$

$$(M_1, M_2, M_3) = (2, 1, 3)$$

$$S = 8/8 = 1$$

$$(3) \qquad r = 1(2, 1, 3) = (x_1, x_2, x_3)$$

Example 2

Solve the system

$$\begin{vmatrix} 2 & -2 & 2 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & -1 & -2 & -3 \\ 3 & -1 & -1 & -1 \end{vmatrix} \begin{vmatrix} x \\ x \\ x \\ x \end{vmatrix} = \begin{vmatrix} -1 \\ 2 \\ -2 \\ 1 \end{vmatrix}$$

This may be written

$$(2) \quad \begin{vmatrix} 2 & -2 & 2 & 1 \\ 5 & -2 & 3 & 4 \\ -5 & 3 & -6 & -5 \\ 5 & -3 & 1 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$(3) \quad h = -25$$

$$(4) \quad (M_1, M_2, M_3, M_4) = (1, 2, 1, -1)$$

$$(5) \quad S = -1/-1 = 1$$

$$(6) \quad r = (1, 2, 1, -1) = (x_1, x_2, x_3, x_4)$$

We consider equation (17), which is the same as equation (21), to be the best of all possible solutions. It is the Mutation Geometry solution of the system (1).

For those of a slightly different taste we continue the developement in the styling of the New Geometry.