Solutions to systems of equations

Example 1.

We first demonstrate the solution of a system of equations arising from a popular electrical engineering problem. Consider the electrical resistive network given below. Resistance values are in ohms. I1, I2, I3, I4, and I5 are loop currents in amps.

```
       1            1            1             1             1
 ___/\/\/\_______/\/\/\________/\/\/\________/\/\/\_______/\/\/\____
 |           |            |             |             |             |
 |           |            |             |             |             |
 |           >            >             >             >             >
----   I1    <      I2    <       I3    <       I4    <      I5     <
 --          >            >             >             >             >
 | 16V       < 11         < 7           < 4           < 2           < 1
 |           |            |             |             |             |
 |           |            |             |             |             |
 -------------------------------------------------------------------
```

Fig. 1

The problem matrix of figure 1 results in a banded matrix. Of the right hand side coefficients, only the first row has a non zero entry.

	I1	I2	I3	I4	I5		V
Row_1	-12	11	0	0	0	=	-16
Row_2	11	-19	7	0	0	=	0
Row_3	0	7	-12	4	0	=	0
Row_4	0	0	4	-7	2	=	0
Row_5	0	0	0	2	-4	=	0

We now form a five-component solution vector

$$G = (u \quad v \quad w \quad x \quad y)$$

When the solution process is complete, u will equal I1, v will equal I2, etc.

The basic idea developed below is to find the values of vector G (u, v, w, x, y) which make G orthogonal to Row_2 through Row_5.

By inspection, we see that we can make vector G orthogonal to Row_5 by setting the x component of G equal to +4 (the I5 coefficient of Row_5 with it's sign reversed) and by setting the y component of G equal to 2 (the I4 coefficient of Row_5 without it's sign reversed). Vector G is now:

$$G = (u \quad v \quad w \quad 4 \quad 2)$$

We observe that vector G is orthogonal to Row_5:

```
                             G * Row_5 = 0
(u  v  w  4  2)*(0  0  0  2  -4) = 0
            0u + 0v + 0w + 8 - 8 = 0
                                   0 = 0
```

To find the w component of G, we set the scalar product of G and Row_4 to zero:

```
                             G * Row_4 = 0
(u  v  w  4  2)*(0  0  4  -7  2) = 0
           0u + 0v + 4w - 28 + 4 = 0
                                   w = 6
```

Vector G is now:

```
G = (u  v  6  4  2)
```

We observe that vector G is orthogonal to Row_4:

```
                             G * Row_4 = 0
(u  v  6  4  2)*(0  0  4  -7  2) = 0
           0u + 0v + 24 - 28 + 4 = 0
                                   0 = 0
```

To find the v component of G, we set the scalar product of G and Row_3 to zero:

```
                              G * Row_3 = 0
(u  v  6  4  2)*(0  7  -12  4  0) = 0
            0u + 7v - 72 + 16 + 0 = 0
                                    v = 8
```

Vector G is now:

```
G = (u  8  6  4  2)
```

We observe that vector G is orthogonal to Row_3:

```
                              G * Row_3 = 0
(u  8  6  4  2)*(0  7 - 12  4  0) = 0
            0u + 56 - 72 + 16 + 0 = 0
                                    0 = 0
```

To find the u component of G, we set the scalar product of G and Row_2 to zero:

```
                               G * Row_2 = 0
(u  8  6  4  2)*(11  -19  7  0  0) = 0
            11u - 152 + 42 + 0 + 0 = 0
                                     u = 10
```

Vector G is now:

```
G = (10  8  6  4  2)
```

We observe that vector G is orthogonal to Row_2:

$$G * Row\_2 = 0$$
$$(10 \;\; 8 \;\; 6 \;\; 4 \;\; 2)*(11 \;\; -19 \;\; 7 \;\; 0 \;\; 0) = 0$$
$$110 - 152 + 42 + 0 + 0 = 0$$
$$0 = 0$$

The components of vector G differ from the solution vector of the system of equations to within a scale factor, K. This scale factor can be determined by requiring that K times the scalar product of vector G and Row_1 of the system matrix equal the right hand side:

$$K(G * Row\_1) = -16$$
$$K(10 \;\; 8 \;\; 6 \;\; 4 \;\; 2)*(-12 \;\; 11 \;\; 0 \;\; 0 \;\; 0) = -16$$
$$K(-120 + 88 + 0 + 0 + 0) = -16$$
$$K = 1/2$$

The solution vector to the system of equations is therefore:

$$G = \tfrac{1}{2}(10 \;\; 8 \;\; 6 \;\; 4 \;\; 2)$$

or

$$G = (5 \;\; 4 \;\; 3 \;\; 2 \;\; 1)$$

Therefore,

$I1 = 5$ amps
$I2 = 4$ amps
$I3 = 3$ amps
$I4 = 2$ amps
$I5 = 1$ amp

Example 2.

Given the following system of equations:

	X1	X2	X3	X4	X5	
Row_1	1	1	1	1	1	= 15
Row_2	1	1	1	1	4	= 30
Row_3	4	2	2	4	3	= 45
Row_4	5	5	3	4	4	= 60
Row_5	7	4	5	5	5	= 75

Using row-column reduction, we set the right sub-diagonal equal to zero as shown below. This leaves, at most, two non-zero coefficients in Row_5.

	X1	X2	X3	X4	X5	
Row_1	1	1	1	1	1	= 15
Row_2	-1	-1	-1	-1	2	= 0
Row_3	1	-1	-1	1	0	= 0
Row_4	1	1	-1	0	0	= 0
Row_5	2	-1	0	0	0	= 0

We now form a five-component solution vector

$$G = (u \quad v \quad w \quad x \quad y)$$

When the solution process is complete, u will equal X1, v will equal X2, etc.

The basic idea developed below is to find the values of vector G (u, v, w, x, y) which make G orthogonal to Row_2 through Row_5.

By inspection, we see that we can make vector G orthogonal to Row_5 by setting the u component of G equal to +1 (the X2 coefficient of Row_5 with it's sign reversed) and by setting the v component of G equal to 2 (the X1 coefficient of Row_5 without it's sign reversed). Vector G is now:

$$G = (1 \quad 2 \quad w \quad x \quad y)$$

We observe that vector G is orthogonal to Row_5:

$$\begin{aligned} G * Row\_5 &= 0 \\ (1 \quad 2 \quad w \quad x \quad y)*(2 \quad -1 \quad 0 \quad 0 \quad 0) &= 0 \\ 2 - 2 + 0w + 0x + 0y &= 0 \\ 0 &= 0 \end{aligned}$$

To find the w component of G, we set the scalar product of G and Row_4 to zero:

$$\begin{aligned} G * Row\_4 &= 0 \\ (1 \quad 2 \quad w \quad x \quad y)*(1 \quad 1 \quad -1 \quad 0 \quad 0) &= 0 \\ 1 + 2 - w + 0x + 0y &= 0 \\ w &= 3 \end{aligned}$$

Vector G is now:

$$G = (1 \;\; 2 \;\; 3 \;\; x \;\; y)$$

We observe that vector G is orthogonal to Row_4:

$$\begin{aligned} G * Row\_4 &= 0 \\ (1 \;\; 2 \;\; 3 \;\; x \;\; y)*(1 \;\; 1 \;\; -1 \;\; 0 \;\; 0) &= 0 \\ 1 + 2 - 3 + 0x + 0y &= 0 \\ 0 &= 0 \end{aligned}$$

To find the x component of G, we set the scalar product of G and Row_3 to zero:

$$\begin{aligned} G * Row\_3 &= 0 \\ (1 \;\; 2 \;\; 3 \;\; x \;\; y)*(1 \;\; -1 \;\; -1 \;\; 1 \;\; 0) &= 0 \\ 1 - 2 - 3 + x + 0y &= 0 \\ x &= 4 \end{aligned}$$

Vector G is now:

$$G = (1 \;\; 2 \;\; 3 \;\; 4 \;\; y)$$

We observe that vector G is orthogonal to Row_3:

$$\begin{aligned} G * Row\_3 &= 0 \\ (1 \;\; 2 \;\; 3 \;\; 4 \;\; y)*(1 \;\; -1 \;\; -1 \;\; 1 \;\; 0) &= 0 \\ 1 - 2 - 3 + 4 + 0y &= 0 \\ 0 &= 0 \end{aligned}$$

To find the y component of G, we set the scalar product of G and Row_2 to zero:

$$\begin{aligned} G * Row\_2 &= 0 \\ (1 \;\; 2 \;\; 3 \;\; 4 \;\; y)*(-1 \;\; -1 \;\; -1 \;\; -1 \;\; 2) &= 0 \\ - 1 - 2 - 3 - 4 + 2y &= 0 \\ y &= 5 \end{aligned}$$

Vector G is now:

$$G = (1 \;\; 2 \;\; 3 \;\; 4 \;\; 5)$$

We observe that vector G is orthogonal to Row_2:

$$\begin{aligned} G * Row\_2 &= 0 \\ (1 \;\; 2 \;\; 3 \;\; 4 \;\; 5)*(-1 \;\; -1 \;\; -1 \;\; -1 \;\; 2) &= 0 \\ -1 - 2 - 3 - 4 + 10 &= 0 \\ 0 &= 0 \end{aligned}$$

The components of vector G differ from the solution vector of the system of equations to within a scale factor, K. This scale factor can be determined by requiring that K times the scalar product of vector G and Row_1 of the system matrix equal the right hand side:

```
                          K(G * Row_1) = 15
      K(1  2  3  4  5)*(1  1  1  1  1) = 15
                  K(1 + 2 + 3 + 4 + 5) = 15
                                     K =  1
```

The solution vector to the system of equations is therefore:

```
      G = 1(1  2  3  4  5)
```

or

```
      G = (1  2  3  4  5)
```

Therefore,

```
      X1 = 1
      X2 = 2
      X3 = 3
      X4 = 4
      X5 = 5
```