Chapter 7

1. Integer Linear Programming (semi-experimental).

We shall first develope a general algorythm for the solution of all integer linear programming problems and then show how to deal with some particular problems which are on a relatively small scale: ones that one may experiment with with pencil and paper and where the general algorythm would require too much computation for such a small prob.

We shall deal with maximum problems first. Maximize the Function

(1) $$P = C \cdot r$$

subject to the restraints

(2) $$A.r = b$$

where all the components of r are required to be non negative integers.

We shall first solve the problem, Mutation-wise, without the integer restriction, obtaining a vector R_0 and a $P_0 = C \cdot R_0$ and an index containing n of the solution equations from the constraint system. When we refer to P_0 subsequently we shall mean the value of P_0 with its decimal part omitted.

(3) Bounding the variables x_j. To get the maximum value of x_1 we change the C in equation (1) to

(4) $$C_1 = 1 + 0 + 0 + 0 + \dots + 0$$

and solve the problem stated above. This solution gives the maximum value of x_1. We then in turn change C in (1) to:

(5) $$C_2 = 0 + 1 + 0 + 0 + \cdots + 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$C_n = 0 + 0 + 0 + 0 + \cdots + 1.$$

and solve the resulting linear programming equations thus giving the maximum value of the variables $x_1 \dots x_n$. In the same way one may solve the same set of equations for the minimum value of the variables $x_1 \dots x_n$. The bounds of our variables limits the number of combinations possible in the general algorithm solution.

In maximization problems one sees from Fig. 24 that the required integer vector r lies somewhere in the feasibility region determined by the constrsint system. One then starts with the equation:

(6) $C \cdot r = P_0$

One then tests this last equation with all the vectors possible under the component bounds calculated above and if any vector is found that satisfies (6) that vector is the answer. If none is found we go with the equation:

(7) $C \cdot r = P_0 - 1$

and repeat the proceds, reducing the right hand term by unity after each failure to find a solution vector, until a solution vector is found. If we were dealing with minimum problems one would increase the right hand term by unity after each failure to find a solution vector. This process assures an answer to all integer linear programming problems. One, to be sure, would need a computer. The scheme above represents some of my early efforts toward integer solutions to linear programming problems, Such a scheme, the general algorithm, is too long for pencil and paper. We devised other schemes. For simple problems one might get by without computing the limits of the variables which represents a lot of work.

2. Numerical solutions of elementary problems in integers.

Minimize, in integers, the function:

(1) $P = (10 + 14 + 21) \cdot r$

subject to the constraints:

(4) $(2 + 2 + 7) \cdot r = 14$

(5) $(9 + 6 + 3) \cdot r = 10$

If this problem is solved without the integer restriction one gets

$r_0 = (28 + 0 + 106)/57$, index (4, 5, 2).

$P_0 = C \cdot r_0 = 44$

We select equation (4) from the index and write it it as:

(4) $(1 + 1 + 3.5) \cdot r = 7$

The possible integer solutioms of (4) that satisfy (5) are

$P (7 + 0 + 0) = 70$

$P (0 + 7 + 0) = 98$

$P (1 + 6 + 0) = 94$

$P (6 + 1 + 0) = 74$

$$P(2 + 5 + 0) = 90$$
$$P(5 + 2 + 0) = 78$$
$$P(3 + 4 + 0) = 86$$
$$P(4 + 3 + 0) = 82.$$

The least value here is 74 but we do not know whether it is the minimum or not. We change the 7 in the last (4) to 8 and get:

$$(4) \qquad (1 + 1 + 2.5) . r = 8$$

whose possible solutions satisfying (4) and (5) are:

$$P(8 + 0 + 0) = 80$$
$$P(0 + 8 + 0) = 112$$
$$P(1 + 0 + 2) = 52$$
$$P(1 + 7 + 0) = 108$$
$$P(7 + 1 + 0) = 84$$
$$P(2 + 6 + 0) = 104$$
$$P(6 + 2 + 0) = 88$$
$$P(3 + 5 + 0) = 100$$
$$P(5 + 3 + 0) = 92$$
$$P(4 + 4 + 0) = 96$$

The least value here is 52. If we change the 8 in the last (4) tO 9 and sweep again we get 62 for the least value of P, showing that 52 is the least value of our function P that satisfies both (4) and (5). We have:

$$P = C . r = (10 + 14 + 21) . (1 + 0 + 2) = 52.$$

What we have done here is to sweep the feasibility reagion near the point obtained without the integer restriction. One could have done the sweeping with equation (5) or in general with any equation in the index of the original equation. We would write equation (5) as:

$$(5) \qquad (9 + 6 + 3) . r = 10 + 5 = 15 \text{ Or}$$

$$(5) \qquad (3 + 2 + 1) . r = 5$$

whose solution is obviously $r = 1 + 0 + 2$ and thus

$$P = C \cdot r = 52$$

which is the minimum. Suppose now that we had operated with the objective function instead of one from the index of the same. In that case we would have startedwith the general algerithm and written initially:

$$P = C. r = (10 + 14 + 21) . r = 44.$$

After computing the limits for the variables, both min. and max., we would have had failures for all values of P from:

$$p = 45, 46, 47, 48, 49, 50, 51$$

and reached success at 52. This would have been a lot of worh in addition to the work of computing the bounds of the variables. The lesson here is obvious for the reader; especially for simple problems. Do not make them complicated. It is good to experiment now and then. It may prove exciting. We do some more experimentation:

3. Gamma Vector Solutions.

Suppose we now take a gamma of equations (4) and (5) of the problem solved in the previous section 2. We get:

(1) $$g = 12 - 19 + 2.$$

In (4) and (5) set $x_3 = 0$ and get

$$x_1 = -32/3, \quad x_2 = 53/3, \quad x_3 = 0.$$

Add $t\, g/3$ to the sum of x_j above and get the solution:

$$r = ((12 t - 32) + (53 - 19 t) + 2 t)/3.$$

In this last equation set $(53 - 19 t) = 0$ or $t = 53/19$ and get:

(2) $$r = (28 + 0 + 106)/57$$

This is the solution without the integer restriction. In this

(3) $$P = C \cdot r = (10 + 14 + 21) . (28 + 0 + 106)/57 = 44.$$

We were a little surprised at the resulting solution. Even tho the answer came out for that of the non integer restriction, one can learn something from it. In semi-experimental solutions there surprises at times. We now rewrite equations (4) and (5) as

(4) $$(2 + 2 + 7) . r = K_1 + 14$$

(5) $$(9 + 6 + 3) . r = K_2 + 10.$$

where K_1 and K_2 are positive constants, small integers hopefully. We now do exactly the same as before in their solution and get:

(6) $r = ((12\,t + 2k_2 - 6\,K_1 - 64) + (9\,K_1 + 106 - 2\,K_2 - 19\,t) + 2\,t)/6.$

Set $9\,K_1 + 106 - 2\,K_2 - 19\,t = 0$ and get:

(7) $r = ((3\,K_1 + 42 - 7\,t) + 0 + 2\,t)/6.$ Set $t = 6$ then

(8) $r = K_1/2 + 0 + 2.$ Set $K_1 = 2$, then

(9) $r = 1 + 0 + 2$

$P = C\,.\,r = 52$

which is the min integer solution.

Perhaps the reader is wondering why we set the quantity, the second component, above equal to 0. We can tell you but you may still wonder about it. It was not by trial and error. I believe what we are going to say is true and that I in the end shall be able to prove it. The reader will notice that the answer to this problem, without the inter restriction, has a 0 for its second component. The answer, without the integer restriction, is:

$r = (28 + 0 + 106)/57$

which has a zero for its sesond component. Does the integer solution of the same problem have a zero for its second component? The accumulative evidence from the solution of several problems seem to say so. On that circumstantial evidence we applied the notion to the present problem and the answer came out the same as by the other methods. The preponderance of evidence seems strong in favor of the idea. We shall use it, and in the meantime try to prove it, till we find a contradiction or some evidence to the contrary. This is experimental mathematics. One can see how much it simplifies matters. In any case all answers have to be tested for satisfaction.

Minimize, in integers, the function

$P = (3 + 4 + 2)\,.\,r$

subject to the constraints

(4) $(1 + 3 - 2)\,.\,r = 6$

(5) $(-3 + 5 + 5)\,.\,r = 15$

(6) $(4 + 2 - 3)\,.\,r = 12$

(7) $(2 + 2 + 5) . r = 10$

(8) $(1 + 3 + 6) . r = 6.$

The solution of this problem, without the integer restriction, was:

$$r_0 = 1.20 + 3.67 + 0.05, \quad \text{index } (5, 6, 7).$$

We now write the index equations as:

(5) $(-3 + 5 + 5) . r = K_1 + 15$

(6) $(4 + 2 - 3) . r = K_2 + 12$

(7) $(2 + 2 + 5) . r = K_3 + 10$

from which we obtain

(9) $x_1 = (-16 K_1 + 15 K_2 + 25 K_3 + 190)/158$

(10) $x_2 = (26 K_1 + 25 K_2 - 11 K_3 + 580)/158$

(11) $x_3 = (-4 K_1 - 16 K_2 + 26 K_3 + 8)/158$

(12) $P = C.r = (48 K_1 + 113 K_2 + 83 K_3 + 2906)/158$

$= (48 K_1 + 113 K_2 + 83 K_3 - 96 + 3002)/158.$

Set $K_1 = 2$, and $K_2 = K_3 = 0$, then $x_1 = 1$, $x_2 = 4$, $x_3 = 0$

$P = 3002/158 = 19 = C . r = (3 + 4 + 2).(1 + 4 + 0).$

In this case there can be no doubt about 19 being the min. ans. since it the first integer above the non- integer solution. The min. vector then is: $r = 1 + 4 + 0$.

In this problem any one of the equations in the index of its original solution could have been used to sweep the feasibility region of the problem. It would be preferable to use the one all of whose coefficients are positive, if such should exist.For small size problems one can obtain the answer withe far less work than by using the general algorithm. Of the numerical solutions the gamma vector solution seem the easiest and require the least computations. They are elegant solutions to the integer requirements. Practice and experiment will suggest other procedures that may be preferable to the general algorithm solution which for even simple problems entail a lot of work for a solution. In any case one can have a variety of solutions from which to choose. It is good to have a choice.